

# Self-Duality and Oblique Confinement in Planar Gauge Theories

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We investigate the non-perturbative structure of two planar  $Z_p \times Z_p$  lattice gauge models and discuss their relevance to two-dimensional condensed matter systems and Josephson junction arrays. Both models involve two compact  $U(1)$  gauge fields with Chern-Simons interactions, which break the symmetry down to  $Z_p \times Z_p$ . By identifying the relevant topological excitations (instantons) and their interactions we determine the phase structure of the models. Our results match observed quantum phase transitions in Josephson junction arrays and suggest also the possibility of *oblique confining ground states* corresponding to quantum Hall regimes for either charges or vortices.

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## 1. Introduction

Topological excitations play a fundamental role in gauge theories with a compact gauge group [1]. Instantons, i.e. topological saddle-point configurations in Euclidean space-time can lead to drastic modifications of the perturbative behaviour of a theory, like confinement and a mass for the gauge fields. Typically, the non-perturbative phase structure of a theory is determined by the condensation (or lack thereof) of such topological configurations in the ground state.

In the case of Abelian gauge theories, the compact  $U(1)$  group can be automatically obtained by spontaneous breakdown of a compact, non-Abelian gauge group. In this case, the ultraviolet cutoff determining the instanton scale is provided by the mass of the gauge fields corresponding to the broken symmetry generators. Alternatively, one can formulate the  $U(1)$  model on a lattice, with the gauge fields being phases of link variables [2]. In this case, the instanton scale is provided by the lattice spacing. This formulation is also particularly suited to study models where the compact  $U(1)$  group is broken down to a discrete gauge group  $Z_p$ .

In (3+1) dimensions,  $Z_p$  lattice gauge theories display a very interesting phase structure [3]. There are two types of string-like topological excitations carrying electric and magnetic quantum numbers respectively. The models are *self-dual* in the sense that the partition function is invariant under the duality transformation exchanging the electric and magnetic excitations and substituting the coupling constant with its inverse. Self-duality is reflected in the phase structure of the models: there is a Higgs phase, when the electric excitations condense in the ground state and a confinement phase when the magnetic excitations condense in the ground state. In these phases, magnetic and electric charges different from multiples of  $p$  are confined, respectively; the mechanism leading to confinement is thus the dual Meissner effect [4]. The phase transition occurs at the self-dual point, where the coupling constant is invariant under the duality transformation. The photon is massive in both phases. For large enough  $p$ , however, the Higgs and confining phases can be separated by a Coulomb phase, in which neither topological excitation condenses in the ground state and the photon is massless.

The complexity of the phase structure is highly increased if a topological  $\theta$ -term is added to the action [5]. In this case, the magnetic excitations carry also electric charge [6]. As a consequence, in addition to the previously described Higgs, confinement and Coulomb phases, we can have new phases characterized by the condensation of topological

excitations carrying both electric and magnetic quantum numbers. In this phases, only particles carrying electric and magnetic quantum numbers in the same ratio as in the condensate emerge as non-confined, physical particles. These phases are therefore called *oblique confinement* phases [7] .

In (2+1) dimensions, the relevant instanton configurations of a compact  $U(1)$  gauge theory are point-like [8] and coincide with the familiar Dirac magnetic monopoles [9] of three-dimensional Minkowski space. These instantons lead to confinement of the fundamental charges of the model and endow the photon with a non-perturbative mass. It is however known [10] [11] that the monopoles are linearly confined themselves, if a topological Chern-Simons term is added to the action. In this case, the relevant topological configurations are string-like: closed strings or open strings with a monopole-antimonopole pair at their ends.

In this paper, we investigate two lattice  $Z_p \times Z_p$  models in (2+1) dimensions <sup>2</sup> which exhibit analogous features to their (3+1)-dimensional counterparts, namely self-duality and oblique confinement. Both models involve two compact  $U(1)$  gauge fields coupled via a mixed Chern-Simons term. Thus, magnetic flux for one gauge field plays the role of the charge coupled to the other. It is this mixed Chern-Simons coupling, which breaks both  $U(1)$  gauge groups down to discrete groups. We study non-perturbative features of these models by identifying the relevant topological configurations and their interactions by a duality transformation [13] . Contrary to the case of Maxwell-Chern-Simons theory, there are no difficulties [14] [11] in formulating a compact lattice gauge model, since the Chern-Simons term is a mixed one.

There are two types of string-like (Euclidean) topological excitations corresponding to the two available charge currents (or magnetic fluxes): contrary to the (3+1)-dimensional  $Z_p$  models, these can be open, in which case they terminate on monopole-antimonopole pairs. These monopoles describe tunneling events leading to the creation (or destruction) of  $p$  localized charges (or magnetic fluxes). The two charges are indeed conserved only modulo  $p$ , due to the discrete gauge symmetry  $Z_p \times Z_p$  [15] . Local gauge invariance is not violated, since the topological excitations couple to the gauge fields only through their curls, which represent topologically conserved currents. The various phases of our models are characterized by the condensation (or lack thereof) of these topological excitations in the ground state. The order parameters distinguishing the different phases are the Wilson

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<sup>2</sup> Related planar  $Z_p$  theories where recently considered in [12]

loop [1] expectation values for the two gauge fields. Contrary to the (3+1)-dimensional models, the photon is massive in all possible phases. This photon mass is a topological one, originating from the mixed Chern-Simons coupling between the two gauge fields.

In addition to their intrinsic field theoretic interest, the models we study are of relevance as effective field theories for two-dimensional condensed matter systems. Indeed, planar gauge fields play an important role in describing the low-energy degrees of freedom for such systems. The key point is that in (2+1) dimensions, a conserved matter current  $j^\mu$  can always be represented in terms of a pseudovector Abelian gauge field  $B_\mu$  as

$$j^\mu \propto \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu . \quad (1.1)$$

This current is usually taken to represent the low-energy matter fluctuations above a given ground-state. The effective theory governing the dynamics of these fluctuations can then be written in terms of  $B_\mu$  as a *gauge theory*. The behaviour of the matter fluctuations is dominated by the lowest dimension term appearing in the gauge field action. Naturally, relativistic invariance does not play any crucial role in these applications. However, relativistic gauge theories provide a framework for studying the relevant physical phenomena, just as the Abelian Higgs model describes the essential features of Landau-Ginzburg effective theories of superconductivity [16] . In these applications to planar condensed matter systems, the gauge symmetry is *compact*, reflecting the underlying lattice structure of the original microscopic model.

The second gauge field in our models lends itself to two possible interpretations. In applications to Josephson junction arrays [17] we take it to encode the vortex dynamics according to an equation analogous to (1.1). Therefore, also the second gauge group is compact. In applications to generic planar condensed matter systems, the second gauge field is taken to describe electromagnetic fluctuations coupled to the low-energy matter excitations. The resulting models are *effective gauge theories*, valid on scales much larger than  $1/\Lambda$ , with  $\Lambda$  the ultraviolet cutoff above which higher-lying matter excitations become important. In order to incorporate the dynamics of magnetic vortices (on scales  $1/\Lambda$ ) in these effective theories, also the electromagnetic gauge field has to be taken as a *compact* variable. The same happens in the Abelian Higgs model, if we neglect completely the radial fluctuations of the Higgs field [18] . Thus, we shall always consider both gauge fields as compact variables.

This paper is organized as follows. In section 2, we formulate the continuum version of the two models in Minkowski space-time and discuss their relevance to planar

condensed matter systems and Josephson junction arrays. In section 3 we introduce our lattice notation, with particular emphasis on the lattice version of the Chern-Simons operator. Sections 4 and 5 are devoted to the lattice formulation of the two models, with compact gauge symmetries, and to the analysis of their phase structure. We shall draw our conclusions in section 6.

## 2. Formulation of the models

### 2.1. Self-dual gauge model in (2+1) dimensions

The first model we consider involves a vector gauge field  $A_\mu$  and a pseudovector gauge field  $B_\mu$  and is defined by the Minkowski space-time Lagrangian (units  $c = 1$  and  $\hbar = 1$ )

$$\begin{aligned}\mathcal{L}_{SD} &= \frac{-1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu + \frac{-1}{4g^2} f_{\mu\nu} f^{\mu\nu} \\ &= \frac{-1}{2e^2} F_\mu F^\mu + \frac{\kappa}{2\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu + \frac{-1}{2g^2} f_\mu f^\mu ,\end{aligned}\tag{2.1}$$

where the field strengths and their duals are given by

$$\begin{aligned}F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu , & F^\mu &\equiv \frac{1}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta} , \\ f_{\mu\nu} &\equiv \partial_\mu B_\nu - \partial_\nu B_\mu , & f^\mu &\equiv \frac{1}{2} \epsilon^{\mu\alpha\beta} f_{\alpha\beta} .\end{aligned}\tag{2.2}$$

Note that the mixed Chern-Simons coupling does not violate the discrete symmetry of parity, due to the pseudovector character of the gauge field  $B_\mu$ .

The coupling constants  $e^2$  and  $g^2$  have dimension mass, whereas the coefficient  $\kappa$  of the mixed Chern-Simons term is dimensionless. In the continuum theory with non-compact gauge fields one could set  $\kappa = 1$  and  $e = g$  by a rescaling of both gauge fields. This is no more possible if the gauge fields are compact variables with a fixed periodicity. This is why we prefer to keep all coupling constants explicit.

The action of the model (2.1) is separately invariant under the two Abelian gauge transformations

$$\begin{aligned}A_\mu &\rightarrow A_\mu + \partial_\mu \lambda , \\ B_\mu &\rightarrow B_\mu + \partial_\mu \omega .\end{aligned}\tag{2.3}$$

The corresponding currents  $f^\mu$  and  $F^\mu$  are topologically conserved. Thus, magnetic flux for one gauge field plays the role of the conserved charge coupled to the other. Moreover, the action corresponding to (2.1) is also invariant under the *duality transformation*

$$\begin{aligned} A_\mu &\longleftrightarrow B_\mu , \\ e &\longleftrightarrow g . \end{aligned} \tag{2.4}$$

The model (2.1) has originally been proposed [19] as an effective theory of planar superconductivity without parity violation. In this application, the conserved current

$$j^\mu \equiv \frac{\kappa}{2\pi} \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu , \tag{2.5}$$

describes *matter* fluctuations about a given superconducting ground-state. These matter fluctuations can be thought of [19] as fermion bound states of an underlying microscopic model. In this case, the coupling constant  $g^2$  sets the scale for the pairing gap  $\Delta$ :  $g^2 = 12\pi\Delta/\kappa^2$ . The last term in (2.1) represents the kinetic term for the matter: in (2+1) dimensions this can be written in terms of the effective gauge field  $B_\mu$  and describes a single, massless scalar field. This is minimally coupled to planar photons, whose kinetic term is given by the first term in (2.1).

The particle content of (2.1) can be easily exposed by the linear transformation

$$\begin{aligned} A_\mu &= \sqrt{\frac{e}{g}} (a_\mu + b_\mu) , \\ B_\mu &= \sqrt{\frac{g}{e}} (a_\mu - b_\mu) . \end{aligned} \tag{2.6}$$

In terms of the new variables  $a_\mu$  and  $b_\mu$ , the Lagrangian (2.1) describes a free theory,

$$\mathcal{L}_{SD} = \frac{-1}{2eg} G_{\mu\nu} G^{\mu\nu} + \frac{\kappa}{2\pi} a_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha a_\nu + \frac{-1}{2eg} g_{\mu\nu} g^{\mu\nu} - \frac{\kappa}{2\pi} b_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha b_\nu , \tag{2.7}$$

where  $G^{\mu\nu}$  and  $g^{\mu\nu}$  are the field strengths for the gauge fields  $a_\mu$  and  $b_\mu$ , respectively. This transformation exposes the mechanism of superconductivity: the original spin 0 and massless photon "absorbs" the matter degree of freedom, thereby turning into a parity and spin ( $\pm 1$ ) doublet with a topological mass [20]

$$m = \frac{|\kappa|eg}{2\pi} . \tag{2.8}$$

As was pointed out in [19], the photon kinetic term has to be modified for potential applications of the model to real quasi-planar high- $T_C$  materials. In these applications, the

dynamics of matter is taken as (2+1)-dimensional, while the electromagnetic field is the real (3+1)-dimensional one. The proper way to describe the coupling of (3+1)-dimensional electromagnetic fields to charges and currents confined to a plane was derived in [21] :

$$\mathcal{L}_{SDP} = \frac{-1}{4e^2} F_{\mu\nu} \frac{1}{\sqrt{\partial^2}} F^{\mu\nu} + \frac{\kappa}{2\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu + \frac{-1}{4g^2} f_{\mu\nu} f^{\mu\nu} . \quad (2.9)$$

Here,  $\partial^2 \equiv \partial_\mu \partial^\mu$  and  $F^{\mu\nu}$  represents the component of the magnetic field perpendicular to the plane and the in-plane components of the electric field. With this modification,  $e^2$  is the usual, dimensionless coupling constant of (3+1)-dimensional electromagnetism. It is easy to convince oneself that (2.9) leads to a  $1/r$  potential between static charges. The effective photon mass of this modified model is easily obtained by integrating out the gauge field  $B_\mu$  and computing the resulting photon propagator:

$$\mu = \frac{\kappa^2 e^2 g^2}{4\pi^2} . \quad (2.10)$$

In addition to its relevance as an effective theory of planar superconductivity, the model (2.1) can also be interpreted as a gauge theory formulation of a planar two-fluid model of coupled charges and vortices. In this application, the gauge field  $A_\mu$  provides an effective description of the vortices via the identification of

$$\Phi^\mu = \frac{\kappa}{2\pi} \epsilon^{\mu\alpha\nu} \partial_\alpha A_\nu \quad (2.11)$$

with the (pseudovector) vortex current. The mixed Chern-Simons term describes then both the Lorentz force exerted by the vortices on the charges and the Magnus force [22] exerted by the charges on the vortices. Indeed, the Magnus force is completely analogous to the Lorentz force: vorticity plays the role of electric charge and fluid density plays the role of the magnetic field.

Charge-charge and vortex-vortex interactions are best exposed in the Coulomb gauge Hamiltonian derived from (2.1). This can be written entirely in terms of the charge and vortex currents  $j^\mu$  and  $\Phi^\mu$  :

$$\begin{aligned} H = \int d^2\mathbf{x} \left\{ j^0 \left( \frac{e^2}{2} \frac{1}{-\nabla^2} + \frac{2\pi^2}{\kappa^2 g^2} \right) j^0 + \Phi^0 \left( \frac{g^2}{2} \frac{1}{-\nabla^2} + \frac{2\pi^2}{\kappa^2 e^2} \right) \Phi^0 \right\} \\ + \int d^2\mathbf{x} \left\{ \frac{2\pi^2}{\kappa^2 g^2} \mathbf{j}_L^2 + \frac{2\pi^2}{\kappa^2 e^2} \Phi_L^2 \right\} , \end{aligned} \quad (2.12)$$

where  $j_L^i$  and  $\Phi_L^i$  denote the longitudinal components of the charge and vortex current densities, respectively. As expected, both the charges and the vortices are subject to long-range Coulomb interactions; there are no charge-vortex interactions other than the Lorentz and Magnus forces mentioned above and these do not contribute to the Hamiltonian. The last two terms in  $H$  represent the kinetic terms for charge and vortex motion, respectively.

The coupled Coulomb gas of charges and vortices described by (2.1) and (2.12) is reminiscent of well known statistical mechanics systems, namely Josephson junction arrays [17]. In order to make further contact with these systems, let us formulate the Hamiltonian (2.12) on a square lattice with lattice spacing  $l$ . To this end we consider the charges and vortices as variables defined on the sites of the lattice (denoted by  $\mathbf{x}$ ), whereas the currents are associated with the links (denoted by  $(\mathbf{x}, i)$ ). Introducing a lattice is actually not sufficient to completely regularize the problem. Indeed, in two spatial dimensions, the lattice Green function  $G(\mathbf{x} - \mathbf{y})$  representing the inverse lattice Laplace operator [23] is still logarithmically divergent for  $\mathbf{x} - \mathbf{y} = 0$  and a further regularization is needed. This leaves the ambiguity of a finite subtraction constant  $\alpha$ . We thus obtain the following lattice Hamiltonian:

$$H_L = \sum_{\mathbf{x}, \mathbf{y}} q_{\mathbf{x}} \frac{e^2}{2} V(\mathbf{x} - \mathbf{y}) q_{\mathbf{y}} + \sum_{\mathbf{x}, \mathbf{y}} \phi_{\mathbf{x}} \frac{g^2}{2} V(\mathbf{x} - \mathbf{y}) \phi_{\mathbf{y}} \\ + \sum_{\mathbf{x}, i} \frac{e^2}{2(ml)^2} q_{\mathbf{x}}^2 v_q^i{}^2 + \sum_{\mathbf{x}, i} \frac{g^2}{2(ml)^2} \phi_{\mathbf{x}}^2 v_{\phi}^i{}^2. \quad (2.13)$$

Here,  $m$  is the topological mass (2.8),  $q_{\mathbf{x}}$  and  $\phi_{\mathbf{x}}$  denote the charge and vortex numbers at site  $\mathbf{x}$  respectively whereas  $v_q^i$  and  $v_{\phi}^i$  label the charge and vortex velocities on the link  $(\mathbf{x}, i)$ . The Green function  $V(\mathbf{x} - \mathbf{y})$  is a lattice kernel with the property  $V(0) = [(1/(ml)^2) - \alpha]$  and behaving as  $-(\log|\mathbf{x} - \mathbf{y}|/2\pi) - \alpha$  at distances large compared to the lattice spacing.

The first two terms in  $H_L$  describe two Coulomb gases of charges and vortices, respectively. The other two terms in  $H_L$  represent kinetic terms for these charges and vortices. They imply the following masses for charges  $q$  and vortices  $\phi$ :

$$m_q = \frac{q^2 e^2}{(ml)^2}, \quad (2.14) \\ m_{\phi} = \frac{\phi^2 g^2}{(ml)^2}.$$

With an appropriate choice of the subtraction constant  $\alpha$  and for  $\kappa = 2$  (representing the charge of Cooper pairs) the Hamiltonian (2.13) reduces essentially to the Hamiltonian of



a Josephson junction array [17] upon identifying the charging energy  $E_C$  and the Josephson coupling  $E_J$  as

$$E_C = \frac{e^2}{4} , \quad E_J = \frac{g^2}{2\pi^2} . \quad (2.15)$$

With this identification, the topological mass (2.8) (for  $\kappa = 2$ ) coincides with the plasma frequency  $\sqrt{8E_CE_J}$  of the array.

The difference between  $H_L$  and the Hamiltonian describing the arrays lies in the kinetic term for the vortices (last term in  $H_L$ ), which is absent in the latter. It is the absence of this term which breaks the perfect duality<sup>3</sup> of (2.13) in the real systems [17]. The vortex kinetic term in our model is connected to the presence of a doublet of propagating degrees of freedom (see (2.7)). Indeed, we could get rid of it by simply projecting out the transverse components of the electric field for  $A_\mu$  from the action and the Hamiltonian. This would leave us with a single propagating degree of freedom of mass (2.8), representing essentially the plasmons in the array. We don't expect the additional vortex kinetic term to induce drastic modifications in the regimes  $e/g \ll 1$  ( $E_C/E_J \ll 1$ ) and  $e/g \gg 1$  ( $E_C/E_J \gg 1$ ), where either charges or vortices clearly dominate the dynamics. However, it is harder to estimate the influence of the additional term in the intermediate region  $e/g \simeq 1$ .

## 2.2. Oblique confining model in (2+1) dimensions

In (3+1) dimensions, the addition of a topological  $\theta$ -term to the action of lattice  $Z_N$  gauge models leads to the appearance of new, oblique confinement phases, characterized by the condensation of topological excitations carrying both electric and magnetic charges [5]. It is therefore natural to investigate if the same phenomenon can take place in (2+1) dimensions. In (2+1) dimensions, the natural topological term to add to the Lagrangian (2.1) is a Chern-Simons term involving only one of the gauge fields, say  $B_\mu$ . However, the model so obtained does not lead to a simple dual (generalized) Coulomb gas representation [13] on a cubic lattice, due to the usual difficulties [14] [11] in inverting the lattice Chern-Simons operator. As we show below, nonetheless, one can get rid of this problem if a further coupling  $F^\mu f_\mu$  is added to the Lagrangian. We consider thus a model defined by the Lagrangian density

$$\mathcal{L}_{OC} = \frac{-1}{2e^2} F_\mu F^\mu + \frac{\kappa}{2\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu - \lambda F_\mu f^\mu + \frac{-1}{2g^2} f_\mu f^\mu + \frac{\eta}{2\pi} B_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu . \quad (2.16)$$

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<sup>3</sup> In terms of the array variables  $E_C$  and  $E_J$ , the self-dual point  $g/e = 1$  is given by  $E_J/E_C = 2/\pi^2$ .

There are two new coupling constants:  $\eta$  is dimensionless, whereas  $\lambda$  has dimension  $\text{mass}^{-1}$ . Both the two new terms violate parity; gauge invariance is clearly maintained, whereas naïve self-duality is broken by the Chern-Simons term.

If we maintain the interpretation (2.5) of  $(\kappa/2\pi)\epsilon^{\mu\alpha\nu}\partial_\alpha B_\nu$  as a conserved current describing matter fluctuations about the ground state of an underlying statistical mechanics model, the additional Chern-Simons term describes a non-local Hopf interaction for this current. The matter degree of freedom is then a topologically massive field [20] of mass  $\eta g^2/\pi$  and spin  $s = \eta/|\eta|$ . The additional coupling  $f_\mu F^\mu$  has the form of a (relativistic) Pauli interaction; correspondingly, the new coupling  $\lambda$  can be viewed as an intrinsic magnetic moment for the matter. In our model, we shall fix this new parameter as follows. By integrating out the electromagnetic gauge field  $A_\mu$ , we obtain an effective theory for the matter degree of freedom:

$$\mathcal{L}_{\text{eff}}^B = -\frac{1}{2} \left( \frac{1}{g^2} - e^2 \lambda^2 \right) f_\mu f^\mu + \frac{e^2 \kappa^2}{8\pi^2} B_\mu \left( \delta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} \right) B_\nu + \frac{\eta - \kappa \lambda e^2}{2\pi} B_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu . \quad (2.17)$$

For generic  $\lambda$ , this theory contains both a Higgs mass and a topological Chern-Simons mass. We shall fix  $\lambda$  by the requirement that the induced Chern-Simons term cancels exactly the bare one, so that only the Higgs mass survives:

$$\lambda = \frac{\eta}{\kappa e^2} . \quad (2.18)$$

For this choice of  $\lambda$ , the interaction with electromagnetic fluctuations is able to lift the frustration in the matter dynamics represented by the current-current Hopf interaction (Chern-Simons term). As we shall show in section 5, it is also this cancellation of the bare and induced Chern-Simons terms, which allows a simple lattice Coulomb gas representation for the topological excitations of the model. With the value (2.18) for  $\lambda$ , (2.17) describes a parity doublet of excitations with spin  $\pm 1$  [24] and mass

$$M = \frac{\frac{eg\kappa}{2\pi}}{\sqrt{1 - \frac{\eta^2 g^2}{\kappa^2 e^2}}} . \quad (2.19)$$

In order to avoid tachyonic excitations, the remaining parameters of the theory must satisfy the condition  $\eta g/\kappa e \leq 1$ .

When (2.18) is satisfied, our model (2.16) is related to the self-dual model introduced in the previous section by a simple transformation of parameters. This is immediately clear, once it is realized that the Lagrangian (2.16) can be rewritten as

$$\begin{aligned} \mathcal{L}_{OC} = & -\frac{1}{2e^2} \left( F_\mu + \frac{\eta}{\kappa} f_\mu \right) \left( F^\mu + \frac{\eta}{\kappa} f^\mu \right) + \frac{\kappa}{2\pi} \left( A_\mu + \frac{\eta}{\kappa} B_\mu \right) \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu \\ & - \frac{1}{2g^2} \left( 1 - \frac{\eta^2 g^2}{\kappa^2 e^2} \right) f_\mu f^\mu . \end{aligned} \quad (2.20)$$

By introducing a new gauge field  $C_\mu$  and a new coupling constant  $g'$  defined as

$$C_\mu \equiv A_\mu + \frac{\eta}{\kappa} B_\mu , \quad g' \equiv \frac{g}{\sqrt{1 - \frac{\eta^2 g^2}{\kappa^2 e^2}}} , \quad (2.21)$$

we recover exactly the self-dual model (2.1). Correspondingly, we can express the mass  $M$  as

$$M = m(e, g') . \quad (2.22)$$

We conclude therefore that our model (2.16) has a *hidden duality symmetry* when (2.18) is satisfied. Note that the new self-dual point  $e/g = \sqrt{1 + \eta^2/\kappa^2}$  lies in allowed range of parameters  $g/e < \kappa/\eta$  for all values of  $\kappa$  and  $\eta$ .

The model (2.16) is also related to known planar condensed matter systems. Indeed, the theory with Lagrangian

$$\mathcal{L}_{CIF} = \frac{\kappa}{2\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu + \frac{\eta}{2\pi} B_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu , \quad (2.23)$$

has been proposed [25] as the effective field theory describing the long distance behaviour of *chiral incompressible fluids* [26] .

The presence of the Chern-Simons term as the dominant kinetic term for matter fluctuations reflects an either explicit or spontaneous breakdown of the discrete  $P$  and  $T$  symmetries in the underlying microscopic model.

For  $2\eta =$  even integer, the effective field theory (2.23) describes the long-distance physics of *chiral spin liquids* [27] ; in this case the  $P$  and  $T$  symmetries are spontaneously broken. For  $2\eta =$  odd integer, the same theory describes the long distance physics of *Laughlin's incompressible quantum fluids*, which are the matter ground states at the plateaus of the quantum Hall effect [28] . In this case, the  $P$  and  $T$  symmetries are explicitly broken by the external magnetic field and  $\nu = 1/2\eta$  plays the role of the filling fraction [29] .

This can be easily seen by integrating out the matter degree of freedom  $B_\mu$ , to obtain an effective action for the gauge potential  $A_\mu$ :

$$S_{\text{eff}}^A = \int d^3x \frac{-\kappa^2}{8\pi\eta} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha A_\nu . \quad (2.24)$$

The induced current is then given by the usual expression

$$j_{\text{in}}^\mu \equiv \frac{\delta}{\delta A_\mu} S_{\text{eff}}^A = -\frac{\kappa^2}{4\pi\eta} \epsilon^{\mu\alpha\nu} \partial_\alpha A_\nu . \quad (2.25)$$

The matter current induced by an electric field  $E^i$  is thus given by

$$j_{\text{in}}^i = \frac{\kappa^2}{2\pi} \frac{1}{2\eta} \epsilon^{ij} E^j , \quad (2.26)$$

which we recognize as the *Hall current* for an incompressible liquid of particles of charge  $\kappa$  (in units of the fundamental charge) and of filling fraction  $1/2\eta$ .

The effective field theory (2.23) describes the incompressible quantum fluids in the limit of an *infinite gap* for matter fluctuations. Indeed, there are no propagating modes, due to the topological nature of both terms in the Lagrangian. Our model (2.16) can be viewed as an extension of (2.23), in which the three possible terms of dimension (mass)<sup>4</sup> coupling the dual field strengths  $F^\mu$  and  $f^\mu$  have been added to the Lagrangian. These can be interpreted as the next-to-leading terms appearing in a derivative expansion of a local, relativistic, gauge invariant effective action for purely planar (also the electromagnetic fluctuations are taken to be (2+1)-dimensional) incompressible fluids. They provide *dynamics* for the gauge fields  $A_\mu$  and  $B_\mu$ , which become propagating degrees of freedom. The resulting topologically massive matter mode represents the so called *magnetophonon* [30] .

With the alternative interpretation of  $(\kappa/2\pi)F^\mu$  as a vortex current, instead, we expect (2.20) to capture the essential physics of Josephson junction arrays in the presence of  $\eta$  external offset charges per plaquette. Correspondingly, had we added a Chern-Simons term for  $A_\mu$ , instead of  $B_\mu$ , we would describe the same systems in presence of an external magnetic field with  $\eta$  fluxes per plaquette. This conjecture, motivated by the analogy with the quantum Hall effect, leads to predictions on the  $T = 0$  phase structure (see section 5) that might be accessible experimentally.

As in the case of the self-dual model of the previous section, (2.16) has to be slightly modified for potential applications to real quantum Hall samples. Specifically, one must

incorporate again (3+1)-dimensional effects and, in particular, a  $1/r$  interaction between charges. Given the representation (2.20), this can be achieved by modifying the model to

$$\mathcal{L}_{OCP} = -\frac{1}{2e^2} \left( F_\mu + \frac{\eta}{\kappa} f_\mu \right) \frac{1}{\sqrt{\partial^2}} \left( F^\mu + \frac{\eta}{\kappa} f^\mu \right) + \frac{\kappa}{2\pi} \left( A_\mu + \frac{\eta}{\kappa} B_\mu \right) \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu - \frac{1}{2g'^2} f_\mu f^\mu , \quad (2.27)$$

with  $e^2$  dimensionless. Note that this modification also changes the logarithmic potential between matter vortices (described by the vortex density  $\epsilon^{ij} \partial_i f_j$ ) to a linear potential. Moreover, also the dynamics of free matter excitations (magnetophonons) is slightly modified. Nonetheless, the limit  $g'^2 \rightarrow \infty$  still describes the limit of an infinite mass.

As emphasized in the introduction, our model does not reproduce the exact dynamics of fluctuations about incompressible quantum fluids; however, it incorporates several essential features of this dynamics which are absent in (2.23), in particular the existence of a *finite gap*

$$\Delta = \mu(e, g') \quad (2.28)$$

for the excitations.

### 3. Lattice Chern-Simons term

As mentioned in the introduction, we would like to investigate the non-perturbative structure of the models described in the previous section, when the gauge fields are compact variables. To this end we shall study the Euclidean partition function of the models on a cubic lattice with lattice spacing  $l$ . Lattice sites are denoted by the vector  $\mathbf{x}$ , and the links between  $\mathbf{x}$  and  $\mathbf{x} + \hat{\mu}$ ,  $\mu = 1, \dots, 3$ , with  $(\mathbf{x}, \mu)$ . The gauge fields  $A_\mu$  and  $B_\mu$  are associated with each link  $(\mathbf{x}, \mu)$ , and for a compact gauge theory they have to be considered as angular variables defined on the interval  $[-\pi/l, \pi/l]$ :

$$\begin{aligned} A_\mu(\mathbf{x}) &\equiv A_\mu(\mathbf{x}) + \frac{2\pi n_\mu(\mathbf{x})}{l} , & n_\mu(\mathbf{x}) &\in Z , \\ B_\mu(\mathbf{x}) &\equiv B_\mu(\mathbf{x}) + \frac{2\pi k_\mu(\mathbf{x})}{l} , & k_\mu(\mathbf{x}) &\in Z . \end{aligned} \quad (3.1)$$

On the lattice, we define the following forward and backward derivatives and shift operators:

$$\begin{aligned} d_\mu f(\mathbf{x}) &\equiv \frac{f(\mathbf{x} + \hat{\mu}l) - f(\mathbf{x})}{l} , & S_\mu f(\mathbf{x}) &\equiv f(\mathbf{x} + \hat{\mu}l) , \\ \hat{d}_\mu f(\mathbf{x}) &\equiv \frac{f(\mathbf{x}) - f(\mathbf{x} - \hat{\mu}l)}{l} , & \hat{S}_\mu f(\mathbf{x}) &\equiv f(\mathbf{x} - \hat{\mu}l) , \end{aligned} \quad (3.2)$$

Summation by parts interchanges both the two derivatives and the two shift operators:

$$\begin{aligned}\sum_{\mathbf{x}} f(\mathbf{x}) d_\mu g(\mathbf{x}) &= - \sum_{\mathbf{x}} \hat{d}_\mu f(\mathbf{x}) g(\mathbf{x}) , \\ \sum_{\mathbf{x}} f(\mathbf{x}) S_\mu g(\mathbf{x}) &= \sum_{\mathbf{x}} \hat{S}_\mu f(\mathbf{x}) g(\mathbf{x}) ,\end{aligned}\tag{3.3}$$

where we have omitted possible surface terms. Gauge transformations are defined by using the forward lattice derivative,

$$A_\mu(\mathbf{x}) \rightarrow A_\mu(\mathbf{x}) + d_\mu \lambda(\mathbf{x}) .\tag{3.4}$$

In order to formulate our models on the lattice, we have to face the problem of defining a lattice version of the Chern-Simons term. This problem has recently received much attention [31] [32] [14] and consists basically in defining a suitable analogue of the Chern-Simons operator  $\epsilon_{\mu\alpha\nu}\partial^\alpha$ . It is easy to verify that this operator is the square root of the familiar Maxwell operator:  $\epsilon_{\mu\gamma\alpha}\partial^\gamma \epsilon^{\alpha\delta\nu}\partial_\delta = -\delta_\mu{}^\nu \partial^2 + \partial_\mu \partial^\nu$ . While in Minkowsky space-time (with discrete space and continuous time) this problem has been solved by Eliezer and Semenoff [14], for the Euclidean version, on a cubic lattice, it turns out that there is no gauge invariant, local operator whose square reproduces the lattice Maxwell operator. In this case we can, however, define the following two lattice operators [31]:

$$K_{\mu\nu} \equiv S_\mu \epsilon_{\mu\alpha\nu} d_\alpha , \quad \hat{K}_{\mu\nu} \equiv \epsilon_{\mu\alpha\nu} \hat{d}_\alpha \hat{S}_\nu ,\tag{3.5}$$

where no summation is implied over equal indices  $\mu$  and  $\nu$ . These operators are both *local* and *gauge invariant*, in the sense that they lead to gauge invariant terms when contracted on both sides <sup>4</sup> with gauge fields:

$$K_{\mu\nu} d_\nu = \hat{d}_\mu K_{\mu\nu} = 0 , \quad \hat{K}_{\mu\nu} d_\nu = \hat{d}_\mu \hat{K}_{\mu\nu} = 0 ,\tag{3.6}$$

The squares of  $K_{\mu\nu}$  and  $\hat{K}_{\mu\nu}$  do not have any particular meaning; however, the product of the two operators reproduces the lattice Maxwell operator,

$$K_{\mu\alpha} \hat{K}_{\alpha\nu} = \hat{K}_{\mu\alpha} K_{\alpha\nu} = -\delta_{\mu\nu} \nabla^2 + d_\mu \hat{d}_\nu ,\tag{3.7}$$

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<sup>4</sup> Note that, on the lattice, gauge invariance requires that kernels are annihilated by  $d_\mu$  on the right and by  $\hat{d}_\mu$  on the left.

where  $\nabla^2 \equiv \hat{d}_\mu d_\mu$  is the three-dimensional, Euclidean Laplace operator on the lattice. In analogy to the forward and backward derivatives and shift operators, also  $K_{\mu\nu}$  and  $\hat{K}_{\mu\nu}$  are interchanged upon summation by parts,

$$\sum_{\mathbf{x},\mu} A_\mu K_{\mu\nu} B_\nu = \sum_{\mathbf{x},\mu} B_\mu \hat{K}_{\mu\nu} A_\nu . \quad (3.8)$$

Both  $K_{\mu\nu}$  and  $\hat{K}_{\mu\nu}$  can be used to define a Chern-Simons term in the lattice action. Hereafter, we choose to use  $K_{\mu\nu}$ .

Using  $K_{\mu\nu}$  we can also define the lattice dual field strengths as

$$\begin{aligned} F_\mu &\equiv K_{\mu\nu} A_\nu , \\ f_\mu &\equiv K_{\mu\nu} B_\nu . \end{aligned} \quad (3.9)$$

These are also compact variables, defined on the interval  $[-\pi/l^2, \pi/l^2]$ . By using (3.8) and (3.7), we easily obtain the following identity:

$$\sum_{\mathbf{x},\mu} F_\mu^2 = \sum_{\mathbf{x},\mu} A_\mu \left( -\delta_{\mu\nu} \nabla^2 + d_\mu \hat{d}_\nu \right) A_\nu , \quad (3.10)$$

which shows that we can write the lattice Maxwell action simply as  $(l^3/2e^2) \sum_{\mathbf{x},\mu} F_\mu^2$ .

## 4. Self-dual model: non perturbative analysis

### 4.1. Lattice formulation and topological excitations

In order to take into account the periodicity of the gauge fields  $A_\mu$  and  $B_\mu$ , we introduce four sets of integer link variables  $\{n_\mu\}$ ,  $\{l_\mu\}$ ,  $\{k_\mu\}$  and  $\{m_\mu\}$ , and we posit the following Euclidean lattice partition function of the Villain type [13]:

$$\begin{aligned} Z &= \sum_{\substack{\{n_\mu\}, \{l_\mu\} \\ \{k_\mu\}, \{m_\mu\}}} \int_{-\frac{\pi}{l}}^{\frac{\pi}{l}} \mathcal{D}A_\mu \mathcal{D}B_\mu \exp(-S) , \\ S &= \sum_{\mathbf{x},\mu} \frac{l^3}{2e^2} \left( F_\mu + \frac{2\pi}{l^2} n_\mu \right)^2 - i \frac{l^3 \kappa}{2\pi} \left( A_\mu + \frac{2\pi}{l} l_\mu \right) K_{\mu\nu} \left( B_\nu + \frac{2\pi}{l} m_\nu \right) \\ &\quad + \frac{l^3}{2g^2} \left( f_\mu + \frac{2\pi}{l^2} k_\mu \right)^2 , \end{aligned} \quad (4.1)$$

where we have introduced the notation  $\mathcal{D}A_\mu \equiv \prod_{\mathbf{x},\mu} dA_\mu(\mathbf{x})$ . This partition function is clearly invariant under the shifts (3.1), since these can be reabsorbed by a redefinition of

the integer link variables. For  $\kappa = 0$ , (4.1) reduces to the sum of two uncoupled copies of the Villain action for compact  $U(1)$  gauge fields studied by Polyakov [1]. In this case, the relevant topological excitations are point-like monopoles, one type for each gauge field. The mixed Chern-Simons coupling between the two gauge fields requires the introduction of two additional integer link variables, in order to maintain the periodicity of the full action.

These additional integer link variables have an important consequence. Using the Poisson summation formula

$$\sum_{k=-\infty}^{+\infty} e^{i2\pi kz} = \sum_{n=-\infty}^{+\infty} \delta(z - n) , \quad (4.2)$$

we recognize that the sums over the integer link variables  $\{m_\mu\}$  and  $\{l_\mu\}$  enforce the following constraints:

$$\begin{aligned} K_{\mu\nu} \left( B_\nu + \frac{2\pi}{l} m_\nu \right) &= \frac{2\pi}{\kappa l^2} \beta_\mu , & \beta_\mu \in Z , \\ \hat{K}_{\mu\nu} \left( A_\nu + \frac{2\pi}{l} l_\nu \right) &= \frac{2\pi}{\kappa l^2} \alpha_\mu , & \alpha_\mu \in Z , \end{aligned} \quad (4.3)$$

for all values of  $A_\mu$ ,  $B_\mu$ ,  $m_\mu$  and  $l_\mu$ . These have the immediate consequence of requiring a quantization condition on the parameter  $\kappa$ :

$$\kappa = p \in Z. \quad (4.4)$$

The integer variables  $\alpha_\mu$  and  $\beta_\mu$  are then identified modulo  $p$ , which means  $\alpha_\mu, \beta_\mu \in Z_p$ .

Let us now consider the variation of the action (4.1) under a gauge transformation  $A_\mu \rightarrow A_\mu + d_\mu \Lambda$ . For simplicity let us take  $\Lambda$  as a function of the first component  $x^1$  only. Under such a gauge transformation the lattice action (4.1) changes by the surface term obtained by summing by parts the second term. Since the boundary conditions are such that the dual field strengths  $F_\mu$  and  $f_\mu$  vanish modulo  $2\pi/l^2$  at infinity, we obtain

$$\Delta S = \sum_{x^2, x^3} -ip \left[ \Lambda(x^1 = +\infty) n_+ - \Lambda(x^1 = -\infty) n_- \right] , \quad (4.5)$$

with  $n_+$  and  $n_-$  integers. Gauge invariance requires that  $\Delta S$  vanishes modulo  $i2\pi$ . This is realized only if  $\Lambda$  takes the values  $\Lambda = (2\pi/p)n$ ,  $n \in Z_p$  at infinity. Clearly, the same holds true for gauge transformations  $B_\mu \rightarrow B_\mu + d_\mu \Lambda$ . This means that both *global*



gauge symmetries are actually broken down to *discrete*  $Z_p$  *symmetries*. An analogous phenomenon has been encountered by Lee [15] in his investigation of continuum, compact Chern-Simons theories.

In the following, we shall investigate how the coupling term affects the topological excitations and their interactions. To this end we decompose  $n_\mu$  and  $k_\mu$  as

$$\begin{aligned} n_\mu &\equiv l K_{\mu\nu} l_\nu + a_\nu , \\ k_\mu &\equiv l K_{\mu\nu} m_\nu + b_\nu , \end{aligned} \tag{4.6}$$

with  $a_\mu$  and  $b_\mu$  integers. The summations over  $\{n_\mu\}$  and  $\{k_\mu\}$  in (4.1) can then be traded for summations over the new integers  $\{a_\mu\}$  and  $\{b_\mu\}$ . Accordingly, we can rewrite the partition function in the following way:

$$\begin{aligned} Z &= \sum_{\substack{\{a_\mu\}, \{l_\mu\} \\ \{b_\mu\}, \{m_\mu\}}} \int_{-\pi}^{\pi} \mathcal{D}A_\mu \mathcal{D}B_\mu \exp(-S) , \\ S &= \sum_{\mathbf{x}, \mu} \frac{l^3}{2e^2} \left[ K_{\mu\nu} \left( A_\nu + \frac{2\pi}{l} l_\nu \right) + \frac{2\pi}{l^2} a_\mu \right]^2 - i \frac{l^3 p}{2\pi} \left( A_\mu + \frac{2\pi}{l} l_\mu \right) K_{\mu\nu} \left( B_\nu + \frac{2\pi}{l} m_\nu \right) \\ &\quad + \frac{l^3}{2g^2} \left[ K_{\mu\nu} \left( B_\nu + \frac{2\pi}{l} m_\nu \right) + \frac{2\pi}{l^2} b_\mu \right]^2 , \end{aligned} \tag{4.7}$$

At this point, we change variables,

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \frac{2\pi}{l} l_\mu , \\ B_\mu &\rightarrow B_\mu + \frac{2\pi}{l} m_\mu , \end{aligned} \tag{4.8}$$

in the integrations over the gauge fields. The sums over the integers  $\{l_\mu\}$  and  $\{m_\mu\}$  can now be carried out explicitly, with the effect of extending the integration interval for the gauge fields from  $[-\pi/l, +\pi/l]$  to  $(-\infty, +\infty)$ :

$$\begin{aligned} Z &= \sum_{\substack{\{a_\mu\} \\ \{b_\mu\}}} \int_{-\infty}^{+\infty} \mathcal{D}A_\mu \mathcal{D}B_\mu \exp(-S) , \\ S &= \sum_{\mathbf{x}, \mu} \frac{l^3}{2e^2} F_\mu^2 - i \frac{l^3 p}{2\pi} A_\mu K_{\mu\nu} B_\nu + \frac{l^3}{2g^2} f_\mu^2 \\ &\quad + \frac{2\pi^2}{le^2} a_\mu^2 + \frac{2\pi^2}{lg^2} b_\mu^2 + \frac{2\pi l}{e^2} A_\mu \hat{K}_{\mu\nu} a_\nu + \frac{2\pi l}{g^2} B_\mu \hat{K}_{\mu\nu} b_\nu . \end{aligned} \tag{4.9}$$

In a last step, we carry out the Gaussian integrations over  $A_\mu$  and  $B_\mu$ . To this end, we introduce the usual gauge fixing terms; these, however, drop out from the final answer, since the gauge fields are coupled to topologically conserved currents  $\hat{K}_{\mu\nu}a_\nu$  and  $\hat{K}_{\mu\nu}b_\nu$ . The result of the Gaussian integrations takes the form  $Z = Z_0 \cdot Z_{\text{Top}}$ , where  $Z_0$  is the lattice partition function for the non-compact, Euclidean version of the model,

$$Z_0 = \int_{-\infty}^{+\infty} \mathcal{D}A_\mu \mathcal{D}B_\mu \exp \sum_{\mathbf{x}, \mu} \left\{ -\frac{l^3}{2e^2} F_\mu^2 + \frac{il^3 p}{2\pi} A_\mu K_{\mu\nu} B_\nu - \frac{l^3}{2g^2} f_\mu^2 \right\} , \quad (4.10)$$

and  $Z_{\text{Top}}$  is given by

$$\begin{aligned} Z_{\text{Top}} &= \sum_{\substack{\{a_\mu\} \\ \{b_\mu\}}} \exp(-S_{\text{Top}}) \\ S_{\text{Top}} &= \sum_{\mathbf{x}, \mu} -\frac{e^2}{2l^3} J_\mu \frac{\delta_{\mu\nu}}{m^2 - \nabla^2} J_\nu - \frac{g^2}{2l^3} K_\mu \frac{\delta_{\mu\nu}}{m^2 - \nabla^2} K_\nu \\ &\quad - i \frac{e^2 g^2 p}{2\pi l^3} J_\mu \frac{K_{\mu\nu}}{\nabla^2(m^2 - \nabla^2)} K_\nu + \frac{2\pi^2}{le^2} a_\mu^2 + \frac{2\pi^2}{lg^2} b_\mu^2 , \end{aligned} \quad (4.11)$$

with the currents  $J_\mu$  and  $K_\mu$  defined by

$$\begin{aligned} J_\mu &\equiv \frac{2\pi l}{e^2} \hat{K}_{\mu\nu} a_\nu , \\ K_\mu &\equiv \frac{2\pi l}{g^2} \hat{K}_{\mu\nu} b_\nu , \end{aligned} \quad (4.12)$$

and the mass  $m$  given in (2.8). In terms of the integer variables  $a_\mu$  and  $b_\mu$ , the action takes its final form

$$\begin{aligned} S_{\text{Top}} &= \sum_{\mathbf{x}, \mu} \frac{2\pi^2}{le^2} a_\mu \frac{m^2 \delta_{\mu\nu} - d_\mu \hat{d}_\nu}{m^2 - \nabla^2} a_\nu + \frac{2\pi^2}{lg^2} b_\mu \frac{m^2 \delta_{\mu\nu} - d_\mu \hat{d}_\nu}{m^2 - \nabla^2} b_\nu \\ &\quad + i \frac{2\pi p}{l} a_\mu \frac{K_{\mu\nu}}{m^2 - \nabla^2} b_\nu . \end{aligned} \quad (4.13)$$

The partition function  $Z_{\text{Top}}$  represents the contribution of the *topological excitations*  $a_\mu$  and  $b_\mu$ , due to the *compactness* of the two gauge symmetries. The string-like excitations  $a_\mu$  and  $b_\mu$  originate as the integer parts of  $F_\mu$  and  $f_\mu$  respectively, and have therefore the obvious interpretation of *magnetic flux strings*. With the interpretation (2.5), however,  $b_\mu$  represents *charge current strings*.

The strings can be closed (rings), in which case  $\hat{d}_\mu a_\mu = 0$  and  $\hat{d}_\mu b_\mu = 0$ , or open, in which case they terminate on monopole-antimonopole pairs. In our Euclidean formalism,

these monopoles describe tunneling events corresponding to the creation or destruction of  $p$  localized, elementary fluxes or charges. Fluxes and charges are indeed conserved only modulo  $p$ , due to the discrete gauge symmetries  $Z_p$ . Note that, in our context, charge does not refer to the particles of the underlying microscopic model; rather it resides on localized, collective *quasi-particle excitations*.

The mechanism leading to string-like topological excitations has its origin in the mixed Chern-Simons coupling <sup>5</sup>. For  $p = 0$  (and therefore  $m = 0$ ),  $S_{\text{Top}}$  reduces to a sum of uncoupled Coulomb gases of monopoles, as expected. Suppose now we start with an isolated monopole for one of the gauge fields: when we turn on the coupling, its otherwise unobservable Dirac string acquires "electric charge" coupled to the other gauge field and becomes thus an observable, physical entity. Moreover, as is evident from (4.13), this charge endows the string with a finite energy per unit length: therefore, infinite open strings do not contribute to the partition function and only closed or finite open strings survive. This means that the mixed Chern-Simons coupling effectively confines the monopoles.

The self-duality of the original model (2.1) is reflected in the invariance of  $S_{\text{Top}}$  under the duality transformation

$$\begin{aligned} a_\mu &\longleftrightarrow b_\mu, \\ e &\longleftrightarrow g. \end{aligned} \tag{4.14}$$

Actually, on the lattice, self-duality is only an approximate symmetry due to the imaginary last term in (4.13), which contains the lattice Chern-Simons operator  $K_{\mu\nu}$ . Under the above duality transformation we obtain an action identical to (4.13), with the only difference that the last term is written in terms of  $\hat{K}_{\mu\nu}$  instead of  $K_{\mu\nu}$ . Note however, that the theories defined with  $K_{\mu\nu}$  and  $\hat{K}_{\mu\nu}$  are completely equivalent.

#### 4.2. Wilson and t'Hooft loops

In order to distinguish the various possible phases of the model we introduce two order parameters, namely the Wilson loop operators [1] for the two gauge fields. Since the gauge field  $B_\mu$  couples to the magnetic flux, the corresponding Wilson loop operator coincides with the magnetic order parameter first introduced by t'Hooft [33] [3]. We shall call it the t'Hooft loop operator. The vacuum expectation values of the Wilson and t'Hooft loop

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<sup>5</sup> The same mechanism is responsible for confinement of monopoles in Maxwell-Chern-Simons theory [11].

operators determine the interaction potential between external test charges and fluxes [1], and provide thus a criterion for confinement.

The lattice version of these operators is given by

$$\begin{aligned} L_W &= \exp iq \sum_{\mathbf{x}, \mu} l q_\mu A_\mu , \\ L_H &= \exp i\phi \sum_{\mathbf{x}, \mu} l \phi_\mu B_\mu , \end{aligned} \quad (4.15)$$

where the integers  $q$  and  $\phi$  represent the strengths of the external test charge and flux, respectively, and  $q_\mu$  and  $\phi_\mu$  vanish everywhere but on the links of the loops, where they take the value 1. Since the loops are closed, they satisfy

$$\hat{d}_\mu q_\mu = \hat{d}_\mu \phi_\mu = 0 . \quad (4.16)$$

We shall concentrate exclusively on the models with the matter fields carrying a charge  $p > 1$ . Correspondingly, we choose  $q, \phi < p$  in our order parameters. It is in fact to be expected that Wilson loops fail as a criterion for confinement for  $p = 1$ , as in the Abelian Higgs model [18].

The computation of the expectation value of the order parameters implies the evaluation of the following integral:

$$\begin{aligned} L &\equiv \langle \exp \sum_{\mathbf{x}, \mu} (iqlq_\mu A_\mu + i\phi l \phi_\mu B_\mu) \rangle \\ &= \frac{1}{Z} \sum_{\substack{\{n_\mu\}, \{l_\mu\} \\ \{k_\mu\}, \{m_\mu\}}} \int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} \mathcal{D}A_\mu \mathcal{D}B_\mu \exp \left\{ -S + \sum_{\mathbf{x}, \mu} (iqlq_\mu A_\mu + i\phi l \phi_\mu B_\mu) \right\} , \end{aligned} \quad (4.17)$$

with  $S$  given in (4.1). Following exactly the same steps as in the evaluation of the partition function, we obtain

$$L = \frac{1}{Z_{\text{Top}}} \sum_{\substack{\{a_\mu\} \\ \{b_\mu\}}} \exp(-W) , \quad (4.18)$$

where  $W = S_{\text{Top}}$  with the currents (4.12) redefined as

$$\begin{aligned} J_\mu &\rightarrow J_\mu + iqlq_\mu , \\ K_\mu &\rightarrow K_\mu + i\phi l \phi_\mu . \end{aligned} \quad (4.19)$$

This leads to the following representation of the order parameter:

$$\begin{aligned}
L &= \exp(-W_0) \frac{1}{Z_{\text{Top}}} \sum_{\substack{\{a_\mu\} \\ \{b_\mu\}}} \exp(-S_{\text{Top}} - W_{\text{Top}}) , \\
W_0 &= \sum_{\mathbf{x}, \mu} \frac{q^2 e^2}{2l} q_\mu \frac{\delta_{\mu\nu}}{m^2 - \nabla^2} q_\nu + \frac{\phi^2 g^2}{2l} \phi_\mu \frac{\delta_{\mu\nu}}{m^2 - \nabla^2} \phi_\nu \\
&\quad - i \frac{2\pi q \phi m^2}{pl} q_\mu \frac{K_{\mu\nu}}{\nabla^2 (m^2 - \nabla^2)} \phi_\nu , \\
W_{\text{Top}} &= \sum_{\mathbf{x}, \mu} -i \frac{2\pi q}{l} q_\mu \frac{\hat{K}_{\mu\nu}}{m^2 - \nabla^2} a_\nu - i \frac{2\pi \phi}{l} \phi_\mu \frac{\hat{K}_{\mu\nu}}{m^2 - \nabla^2} b_\nu \\
&\quad - qe^2 p A_\mu^q \frac{K_{\mu\nu}}{m^2 - \nabla^2} b_\nu - \phi g^2 p A_\mu^\phi \frac{\hat{K}_{\mu\nu}}{m^2 - \nabla^2} a_\nu ,
\end{aligned} \tag{4.20}$$

where we have introduced the elementary area elements  $A_\mu^q$  and  $A_\mu^\phi$  of the closed loops:

$$\begin{aligned}
q_\mu &= l \hat{K}_{\mu\nu} A_\nu^q , \\
\phi_\mu &= l \hat{K}_{\mu\nu} A_\nu^\phi .
\end{aligned} \tag{4.21}$$

These vanish everywhere but on the links perpendicular to the elementary plaquettes spanning the minimal area enclosed by the loops, where they take the value 1.

In (4.20), the factor  $\exp(-W_0)$  represents the contribution from the massive, propagating modes described by the partition function  $Z_0$  in (4.10). The second factor, instead, describes the contribution from the topological excitations.

The first two terms in  $W_0$  describe the screened Coulomb interaction mediated by the massive gauge particles. The third term, instead, represents the *Aharonov-Bohm interaction* between charges  $q$  and fluxes  $\phi$  separated on distances much larger  $1/m$ . In this case one can in fact neglect all the off-diagonal terms in the kernel  $1/(m^2 - \nabla^2)$  and retain only its diagonal term  $1/m^2$ , thereby obtaining the expression

$$\sum_{\mathbf{x}, \mu} q_\mu \frac{l K_{\mu\nu}}{l^2 \nabla^2} \phi_\nu . \tag{4.22}$$

This is an *integer*, as can be easily recognized by inserting the representation (4.21) for  $\phi_\mu$ . This integer is a lattice version of the *Gauss linking number* of the two closed loops. The Aharonov-Bohm interaction vanishes thus for  $q\phi = np$ ,  $n \in \mathbb{Z}$ , which is the celebrated Dirac quantization condition (in our units) for a discrete  $Z_p$  gauge theory.

The contribution  $W_{\text{Top}}$  vanishes when both types of topological excitations are suppressed, i.e. when the partition function  $Z_{\text{Top}}$  is dominated by the saddle point  $a_\mu = 0$ ,  $b_\mu = 0$ . In this case, nothing is changed with respect to the above picture. Charges and fluxes come in neutral bound states with binding energies of order  $\log(1/ml)$  and are essentially free for  $ml > O(1)$ . The photon has a *topological mass* due to the Chern-Simons mechanism. We shall therefore call this phase of the theory the *Chern-Simons phase*. For  $ml > O(1)$  it can be identified with the mixed phase of a type-II superconductor. This Chern-Simons phase corresponds to the Coulomb phase of (3+1)-dimensional  $Z_N$  gauge models. In (2+1) dimensions a pure Coulomb phase does not exist since the photon is always massive due to the Chern-Simons mechanism. It is this phase of the model which was investigated in [19].

This picture can be drastically changed when one type of topological excitations condenses. To see this let us suppose that  $Z_{\text{Top}}$  is dominated by the saddle point  $b_\mu = 0$ , while the formation of long  $a_\mu$  strings is favoured. There are two types of strings: open ones and closed ones. For the planar quantum system the former describe tunneling events corresponding to the formation and subsequent destruction of localized fluxes  $p$ ; the latter describe instead the formation and annihilation of neutral pairs of such fluxes. The condensation of long strings in the three-dimensional statistical mechanics problem indicates that the ground state of the planar quantum system consists of a magnetic condensate. The total flux number of the ground state fluctuates around zero due to the monopoles at the end of the open strings. We now show that such fluctuations confine electric charges.

First of all let us remark that a condensation of long  $a_\mu$  strings is accompanied by a condensation of closed strings  $\hat{K}_{\mu\nu}a_\nu$ , representing circular electric currents. These couple to external test fluxes  $\phi$  via the last term in  $W_{\text{Top}}$ . However, these electric currents form *tiny* loops around the long magnetic flux lines  $a_\mu$  and their effects on test fluxes are essentially negligible [1] (apart from a renormalization of the coupling constant of the screened Coulomb potential).

The only relevant term in  $W_{\text{Top}}$  is therefore the first term, coupling  $a_\mu$  to the Wilson loop  $q_\mu$ . For  $ml \gg 1$ , we can neglect  $\nabla^2$  in the interaction kernel  $1/(m^2 - \nabla^2)$ . In this case, the coupling reduces to a contact term between the magnetic flux string  $a_\mu$  and the tiny flux rings  $K_{\mu\nu}q_\nu$  encircling the Wilson loop  $q_\mu$ . This contact term will contribute only a perimeter law to the Wilson loop expectation value and essentially renormalizes the first term in  $W_0$ . The same argument can be repeated when the electric strings  $b_\mu$  condense: we thus conclude that for  $ml \gg 1$  the system possesses only the Chern-Simons phase.

Let us now concentrate on the case  $ml \ll 1$  and let us consider Wilson loops with typical dimension  $L$  in the range  $l \ll L \ll 1/m$ . For such loops we can neglect  $m^2$  in the interaction kernel with all strings  $a_\mu$  (first term in  $W_{\text{Top}}$ ) passing through the surface spanned by the loop. As a consequence, these strings become unobservable for the Wilson loop, which couples only to the monopoles at their end:

$$\sum_{\mathbf{x}, \mu} i \frac{2\pi q}{l} q_\mu \frac{\hat{K}_{\mu\nu}}{\nabla^2} a_\nu = \sum_{\mathbf{x}} -i \frac{2\pi q}{l} \hat{d}_\mu A_\mu^q \frac{1}{\nabla^2} Q, \quad (4.23)$$

since  $A_\mu^q a_\mu$  is an integer. Here  $Q \equiv l \hat{d}_\mu a_\mu$  represents the monopoles. This means that only the longitudinal degrees of freedom of  $a_\mu$  couple to the Wilson loop. As it is easy to see by inserting the representation

$$a_\mu = l d_\mu \omega, \quad l^2 \nabla^2 \omega = Q, \quad (4.24)$$

into  $S_{\text{Top}}$ , these describe a Coulomb gas of magnetic monopoles. Charges are therefore confined by the familiar Polyakov mechanism [1].

Actually, the above computation indicates only the presence of a linear potential between charges up to scales  $1/m$ . However, since the string tension is of order  $e^2/l$  [1], the energy required to separate two charges is at least of order  $e^2/(ml) = (2\pi e/pg)(1/l)$ . In the next section we shall show that the condensation of magnetic strings is favoured for large values of  $e/g$ . The binding energy is therefore much larger than the ultraviolet cutoff  $\Lambda = 1/l$  and charges are effectively confined.

We thus conclude that the phase in which the condensation of magnetic strings  $a_\mu$  is favoured is a *confinement (or insulating) phase*. The same arguments repeated for strings  $b_\mu$  lead to the corresponding conclusion that the phase in which the condensation of electric strings  $b_\mu$  is favoured is a *Higgs (or superconducting) phase*.

#### 4.3. Phase structure analysis

As explained above, in order to establish the phase structure of the model as a function of its parameters, we need to analyze the conditions for the condensation of the topological excitations. To this end, we shall use the same free energy arguments adopted in the analysis of the related (3+1)-dimensional models [3][5]. In these arguments, the condition for condensation of strings is established by analyzing the balance between the self-energy of a string and its entropy.

The free energy of a string of length  $L = lN$  carrying magnetic and electric quantum numbers  $a$  and  $b$  is essentially

$$\beta F = \left( \frac{2\pi^2}{le^2} (ml)^2 G(ml) a^2 + \frac{2\pi^2}{lg^2} (ml)^2 G(ml) b^2 - \gamma \right) N , \quad (4.25)$$

where  $G(ml)$  is the diagonal element of the lattice kernel  $G(\mathbf{x}-\mathbf{y})$  representing the inverse of the operator  $l^2(m^2 - \nabla^2)$ . Clearly, this diagonal element also depends on the dimensionless parameter  $ml$ . The last term in (4.25) represents the entropy of the string: the parameter  $\gamma$  is given roughly by  $\gamma = \ln 5$ , since at each step the string can choose between 5 different directions. In a dilute instanton approximation, in which all values  $a_\mu, b_\mu \geq 2$  are neglected, it can be proved that the correct value of  $\gamma$  is the same for open and closed strings [34]. In (4.25) we have neglected all subdominant functions of  $N$ , like a  $\ln N$  correction to the entropy and a constant term due to the monopole contribution to the energy for open strings. Moreover, we have neglected the imaginary term in the action (4.13). This can be justified self-consistently, since the contribution of this term vanishes in all phases of the model, as we now show.

The condition for the condensation of topological excitations is obtained by minimizing the free energy (4.25) as a function of  $N$ . If the coefficient of  $N$  in (4.25) is positive, the minimum of  $\beta F$  is obtained for  $N = 0$  and topological excitations are suppressed. If instead the same coefficient is negative, the minimum of  $\beta F$  is obtained for  $N = \infty$  and the system will favour the formation of long strings. Topological excitations with quantum numbers  $a$  and  $b$  condense therefore if

$$\frac{2\pi^2}{le^2\delta} a^2 + \frac{2\pi^2}{lg^2\delta} b^2 < 1 , \quad (4.26)$$

where we have introduced

$$\delta \equiv \frac{\gamma}{(ml)^2 G(ml)} . \quad (4.27)$$

This new parameter is clearly also a function of  $ml$ . When two or more condensates are possible, one has to choose the one with the lowest free energy.

This condensation condition describes the interior of an ellipse with semi-axes  $le^2\delta/2\pi^2$  and  $lg^2\delta/2\pi^2$  on a square lattice of integer magnetic and electric charges. The phase diagram is obtained by investigating which points of the integer lattice lie inside the ellipse as its semi-axes are varied. We find it convenient to present the result in terms of the



parameters  $lm$  and  $e/g$ . For  $lm \gg 1$  we have only the Chern-Simons phase, for all values of  $e/g$ . For  $ml \ll 1$  we obtain instead the following phase structure:

$$\begin{aligned} \frac{\delta lm}{\pi p} > 1 &\rightarrow \begin{cases} \frac{e}{g} < 1, & \text{Higgs (superconducting)} , \\ \frac{e}{g} > 1, & \text{confinement (insulating)} , \end{cases} \\ \frac{\delta lm}{\pi p} < 1 &\rightarrow \begin{cases} \frac{e}{g} < \frac{\delta lm}{\pi p}, & \text{Higgs (superconducting)} , \\ \frac{\delta lm}{\pi p} < \frac{e}{g} < \frac{\pi p}{\delta lm}, & \text{Chern-Simons} , \\ \frac{e}{g} > \frac{\pi p}{\delta lm}, & \text{confinement (insulating)} . \end{cases} \end{aligned} \quad (4.28)$$

As expected, the phase diagram is symmetric around the self-dual point  $e/g = 1$ . For small  $e/g$  we obtain a Higgs (superconducting) phase, while for large  $e/g$  the model is in a confinement (insulating) phase. However, for  $\delta lm/\pi p < 1$ , these phases do not extend all the way to  $e/g = 1$ ; rather, an intermediate Chern-Simons phase opens up between the Higgs and confinement phases. The results (4.28) were derived assuming  $ml \ll 1$ . The presence or absence of an intermediate Chern-Simons phase depends therefore on the exact form of the function  $lm\delta(lm)$  for  $lm \ll 1$ .

We conclude this section by stressing that an insulating-superconducting quantum phase transition is actually observed experimentally [35] in planar Josephson junction arrays at extremely low temperatures. This further confirms that the self-dual lattice gauge theory (4.1) (for  $\kappa = 2$ ) captures the essential physics of planar Josephson junction arrays and raises the question whether the intermediate Chern-Simons phase might be experimentally accessible at even lower temperatures.

#### 4.4. Including (3+1)-dimensional effects

Up to now we have discussed only purely planar effects. As we pointed out in section 2, however, the photon kinetic term has to be modified as in (2.9) in order to describe (3+1)-dimensional electromagnetism coupled to planar matter. In the following we investigate how this modification affects the phase structure of the model.

The new lattice model is easily obtained by substituting the first term in the action (4.1) with

$$\frac{l^3}{2e^2} \left( F_\mu + \frac{2\pi}{l^2} n_\mu \right) \frac{1}{\sqrt{-\nabla^2}} \left( F_\mu + \frac{2\pi}{l^2} n_\mu \right) , \quad (4.29)$$

where  $e^2$  is now dimensionless. All the steps leading to (4.13) and (4.20) can be exactly repeated. The resulting modified expressions for  $S_{\text{Top}}$ ,  $W_0$  and  $W_{\text{Top}}$  are given by:

$$\begin{aligned}
S_{\text{Top}} &= \sum_{\mathbf{x}, \mu} \frac{2\pi^2}{le^2} a_\mu \frac{\mu\sqrt{-\nabla^2}\delta_{\mu\nu} - d_\mu \hat{d}_\nu}{-\nabla^2 (\sqrt{-\nabla^2} + \mu)} a_\nu + \frac{2\pi^2}{lg^2} b_\mu \frac{\mu\sqrt{-\nabla^2}\delta_{\mu\nu} - d_\mu \hat{d}_\nu}{\sqrt{-\nabla^2} (\sqrt{-\nabla^2} + \mu)} b_\nu \\
&\quad + i \frac{2\pi p}{l} a_\mu \frac{K_{\mu\nu}}{\sqrt{-\nabla^2} (\sqrt{-\nabla^2} + \mu)} b_\nu , \\
W_0 &= \sum_{\mathbf{x}, \mu} \frac{q^2 e^2}{2l} q_\mu \frac{\delta_{\mu\nu}}{\sqrt{-\nabla^2} + \mu} q_\nu + \frac{\phi^2 g^2}{2l} \phi_\mu \frac{\delta_{\mu\nu}}{\sqrt{-\nabla^2} (\sqrt{-\nabla^2} + \mu)} \phi_\nu \\
&\quad - i \frac{2\pi q \phi \mu}{pl} q_\mu \frac{K_{\mu\nu}}{\nabla^2 (\sqrt{-\nabla^2} + \mu)} \phi_\nu , \\
W_{\text{Top}} &= \sum_{\mathbf{x}, \mu} -i \frac{2\pi q}{l} q_\mu \frac{\hat{K}_{\mu\nu}}{\sqrt{-\nabla^2} (\sqrt{-\nabla^2} + \mu)} a_\nu - i \frac{2\pi \phi}{l} \phi_\mu \frac{\hat{K}_{\mu\nu}}{\sqrt{-\nabla^2} (\sqrt{-\nabla^2} + \mu)} b_\nu \\
&\quad - qe^2 p A_\mu^q \frac{K_{\mu\nu}}{\sqrt{-\nabla^2} + \mu} b_\nu - \phi g^2 p A_\mu^\phi \frac{\hat{K}_{\mu\nu}}{\sqrt{-\nabla^2} (\sqrt{-\nabla^2} + \mu)} a_\nu ,
\end{aligned} \tag{4.30}$$

where the mass  $\mu$  is given in (2.10).

In this model, the Chern-Simons phase (characterized by the absence of topological excitations) consists of charges interacting via a screened  $1/r$ -interaction (Coulomb interaction in (3+1) dimensions) and fluxes interacting via a logarithmic potential (Coulomb interaction in (2+1) dimensions) up to scales  $1/\mu$  and a  $1/r$ -interaction on larger scales. The photon is still massive and it is this photon mass which screens the 2- and 3-dimensional Coulomb interactions on scales  $1/\mu$ . Note that the Aharonov-Bohm interaction for charges and fluxes separated by distances much larger than  $1/\mu$  is unaffected by the modification (4.29).

Let us now consider again the effects of the condensation of topological excitations. Suppose first that the electric strings  $b_\mu$  condense. In this case one can repeat verbatim the analysis of the preceding section with the same conclusion that this is a Higgs (superconducting) phase. When the magnetic strings  $a_\mu$  condense one can also repeat the above analysis; however in this case there is a crucial difference with respect to the purely planar case. The Wilson loop still couples only to the longitudinal degrees of freedom of  $a_\mu$ , which are represented by the magnetic monopoles. However, using the representation (4.24) in the first term of  $S_{\text{Top}}$  we obtain

$$S_Q = \sum_{\mathbf{x}} \frac{2\pi^2}{e^2 l^3} Q \frac{1}{-\nabla^2 \sqrt{-\nabla^2}} Q . \tag{4.31}$$

This is the Hamiltonian for magnetic monopoles with a logarithmic interaction (at large distances) in 3 dimensions. The logarithmic interaction is confining. The same free energy arguments used to derive the phase structure of a two-dimensional Coulomb gas suggest the existence of a strong coupling phase at low  $e^2$ , in which the monopoles are confined. We expect this phase to be realized for the small value of the fine structure constant  $e^2/4\pi$ . Since monopoles are confined, they cannot screen the dipole sheet  $\hat{d}_\mu A_\mu^q$  in (4.23) and the Wilson loop does not acquire an area law. This means that for a sufficiently weak Coulomb interaction of charges in the model (2.9) there is no confinement phase, which is the expected result.

The transition point between the Chern-Simons phase and the Higgs (superconducting) phase is determined by the condition for condensation of the electric strings  $b_\mu$ . In analogy to (4.26) this is given by

$$\frac{p^2 e^2}{2\delta(\mu l)} b^2 < 1 . \quad (4.32)$$

Here  $\delta(\mu l) \equiv \gamma/G(\mu l)$ , and  $G(\mu l)$  is the diagonal element of the lattice kernel representing the inverse of  $l(\sqrt{-\nabla^2} + \mu)$ . We thus obtain the following phase structure:

$$\begin{aligned} \delta(\mu l) &> \frac{p^2 e^2}{2} , & \text{Higgs (superconducting) ,} \\ \delta(\mu l) &< \frac{p^2 e^2}{2} , & \text{Chern - Simons .} \end{aligned} \quad (4.33)$$

The function  $\delta(\mu l)$  has the following asymptotic behaviour,

$$\begin{aligned} \mu l &\rightarrow 0 , & \delta(\mu l) &= \text{const.} , \\ \mu l &\rightarrow \infty , & \delta(\mu l) &\propto \mu l . \end{aligned} \quad (4.34)$$

This implies that the system is always in the Higgs (superconducting) phase for a sufficiently large mass gap  $\mu$  (pairing gap in the underlying microscopic model [19]). The possible presence of the additional Chern-Simons phase at  $T = 0$  depends again on the details of the function  $\delta(\mu l)$ .

## 5. Oblique confining model: non-perturbative analysis

In this section we shall consider the models with a topological Chern-Simons term for the matter gauge field  $B_\mu$ .

Formulating a compact lattice version of (2.20) and (2.27) requires again the introduction of integer link variables, which enforce constraints analogous to (4.3). These require the quantization of both parameters  $\kappa$  and  $\eta$ :

$$\kappa = p \in Z , \quad \eta = n \in Z . \quad (5.1)$$

The ensuing gauge group depends crucially on the commensurability of  $p$  and  $n$ . If  $p$  and  $n$  are coprime, the original  $U(1) \times U(1)$  global gauge group is completely broken. If, instead,  $p$  and  $n$  have a (maximal) common factor  $r$ , the residual discrete gauge symmetry is  $Z_r \times Z_r$ .

Given the representations (2.20) and (2.27), we can just make use of the results of the previous section, provided we make the following substitutions:

$$\begin{aligned} g &\rightarrow g' , \\ m &\rightarrow M , \\ \mu &\rightarrow \Delta , \\ a_\mu &\rightarrow \left( a_\mu + \frac{n}{p} b_\mu \right) , \\ \phi &\rightarrow \left( \phi + \frac{n}{p} q \right) . \end{aligned} \quad (5.2)$$

We still have a Chern-Simons phase and, for the purely planar model, a confinement phase; however the third possible phase changes completely its character. In the model of the previous section, this third phase was characterized by the condensation of electric strings  $b_\mu$ , while  $a_\mu = 0$ . With the above substitutions, this means a condensation of strings carrying both *electric and magnetic quantum numbers*  $a$  and  $b$  in the ratio  $a/b = -n/p$ , i.e. of dyonic strings. Correspondingly, this condensation implies that particles with quantum numbers  $\phi + (n/p)q \neq 0$  are confined. The ground state of the planar quantum system consists of a dyonic condensate; excitations carry both electric charge and magnetic flux in the same ratio as in the condensate, i.e.  $\phi + (n/p)q = 0$ .

If  $p = rP$  and  $n = rN$ , with  $P$  and  $N$  coprime, the elementary excitations carry fractional (in units of the fundamental charge  $p$ ) charge  $P/p$  and magnetic flux  $N$ . These excitations are *anyons* [36] with fractional statistics  $PN/p$  (modulo 2) originating in the Aharonov-Bohm interaction of an elementary charge  $P$  with the flux  $N$  carried by a second elementary charge.

This phase of the model is an *oblique confinement* phase [7] [5]. It is characterized by a gap, the absence of longitudinal conductivity and the presence of a quantized Hall conductivity. We thus identify the corresponding ground state of the system as Laughlin's *incompressible quantum fluid* [37].

We expect this phase to be realized when all offset charges (external magnetic fluxes for the model in (3+1) dimensions) can be attached to the vortices (charges for the model in (3+1) dimensions), i.e. for  $n = p$ . In this case, the fractional charge and statistics of elementary excitations are both given by  $1/p$ .

In (2.26) we have derived the Hall current characterizing the oblique confinement phase (for  $n = p$ ) as

$$j_H^i = \frac{(p/2)^2}{2\pi} \frac{1}{(p/2)} \epsilon^{ij} E^j . \quad (5.3)$$

For the smallest allowed value of  $p$ , the matter gauge fields describe therefore particles of charge  $1/2$ . If we don't want to describe physical electrons as Cooper pairs of charge  $1/2$  particles we must rescale all charges by a factor 2 and therefore  $p \rightarrow 2p$ ,  $n \rightarrow 2n$ . Moreover,  $p$  has to be odd if electrons are to retain their fermionic character. In the resulting description, the physical electron is identified with a particle of charge  $p$ : all charges  $1 \dots p$  represent fractional charge particles. As a consequence of the original compact gauge symmetry, all charges are quantized in integer units: fractional Hall states are thus described by increasing the charge of the electron.

Correspondingly, we describe bosonic Cooper pairs by choosing  $p$  even (after the above rescaling). This is the relevant situation for applications to Josephson junction arrays. As we showed in (2.17), by integrating out the vortex degrees of freedom one obtains an effective theory for the charges which does not contain any Chern-Simons term. Conversely, if we integrate out the charge degrees of freedom we obtain an effective action for the vortices which contains a Chern-Simons term with coefficient  $p/4\pi$  (after the above rescaling). Repeating the computation leading to (2.26), we recognize that the oblique confinement phase is a quantum Hall regime for the vortices. Had we added originally a Chern-Simons term for the  $A_\mu$  gauge field, we would obtain correspondingly a quantum Hall regime for the charges. In this case, the Hall conductivities would take the form

$$\sigma_H = \frac{(2e)^2}{2\pi} \frac{1}{p} , \quad p = \text{even} . \quad (5.4)$$

In the purely planar model, the oblique confinement phase described above is realized for small  $e/g'$  ( $E_C/E_J$ ). In this context,  $g'^2$  has to be understood as the physical, renormalized (by the external offset charges) Josephson coupling. As we explained above, this

corresponds to a quantum Hall regime for the vortices. Above a critical value for  $E_C/E_J$ , the system undergoes a transition to a confinement (insulating) phase. Again, the presence of a possible intermediate Chern-Simons phase depends on the detailed behaviour of the function  $lM\delta(Ml)$ . In presence of external magnetic fields, correspondingly, we obtain a quantum Hall regime for charges for large values of  $e'/g$  ( $E_C/E_J$ ). In this case, the system is in a Higgs (superconducting) phase for small values of the same parameter.

In the model including (3+1)-dimensional effects, we obtain a flux unbinding transition from the oblique confinement phase (quantum Hall regime) to a Chern-Simons phase when the gap  $\Delta$  is lowered below a critical value. Due to the  $1/r$  interactions among charges in this latter phase, it is to be expected that this phase immediately cristallizes at the low temperatures in which quantum Hall experiments are performed. Note that the critical gap is an increasing function of  $p$ , as is evident from (4.33). This explains the lesser stability of quantum Hall states with smaller filling fraction.

It is known [38] that the microscopic Laughlin wave functions [37] for the incompressible quantum fluids can be viewed as quantum states in which an odd number of statistical fluxes are bound to the electrons. This fact is at the basis of Jain's theory [39] of composite electrons and of most field theoretic treatments [40] of the quantum Hall effect. Our results demonstrate how the key aspect of the quantum Hall effect is oblique confinement by the Polyakov monopole mechanism. Given the non-perturbative nature of our treatment we could explicitly derive the existence of a *critical gap* for the formation of incompressible quantum fluids. We believe that this is a new and important result in the framework of effective field theories for the quantum Hall effect.

## 6. Concluding remarks

We would like to conclude this paper with the following two observations. First, our results suggest that Josephson junction arrays might provide an easily accessible experimental setting for testing "Chern-Simons physics" and most characteristic phenomena in planar gauge theories, like topological photon masses, fractional statistics and Polyakov's confinement mechanism. Secondly, the actual observation of the oblique confinement phase of the theory (2.27) in quantum Hall experiments suggests that its "sister theory" (2.9) is indeed a strong candidate for an effective field theory of quasi-planar superconductivity. Indeed, the physical mechanism leading to superconductivity in this latter model is exactly the same mechanism which is responsible for the formation of the quantum Hall fluids in the former model.

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